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PRINCIPAL STRUCTURE OF SUBMODULAR SYSTEMS AND HITCHCOCK-TYPE INDEPENDENT FLOWS

SATORU IWATA

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This paper discusses the principal structure of submodular systems due to S. Fujishige. It is shown that the principal structure is the coarsest decomposition that is finer than any decomposition induced by the principal partition with respect to a minimal nonnegative superbase. The concept of "Hitchcock-type independent flow" is introduced so that previously known results on the principal structures for bipartite matchings, layered mixed matrices and independent matchings can be understood as applications of the present result.

1. Introduction

In the past couple of decades, the theory of matroids has become widely recognized to be useful for practical problems in engineering as can be seen in Iri [8] and Recski [17]. A *submodular system* is a pair of distributive lattice and a submodular function on it and is a generalization of a matroid. Efficient algorithms in matroid theory, such as the greedy algorithm and matroid intersection algorithms, have naturally been extended to submodular systems.

From the practical point of view, however, it is important not only to obtain an optimal solution but also to grasp hierarchical structure by an appropriate decomposition technique. Thus the concept of *principal partition* has been introduced. The principal partition is based on the Jordan-Hölder type decomposition principle expounded in Iri [9], and is an extension of the Dulmage–Mendelsohn decomposition [2] for bipartite graphs.

One of the applications of the principal partition is the combinatorial canonical form of layered mixed matrices, which has been proposed by Murota [13] as a mathematical tool for describing discrete physical/engineering systems. It is known that there uniquely exists a finest block-triangularization of an LM-matrix, which is called the *combinatorial canonical form* (or *CCF*). See also [15, 16] for LM-matrices.

Another decomposition principle, named *principal structure* of a submodular system has been proposed by Fujishige [5] in a somewhat abstract context. An

Mathematics Subject Classification (1991): 05 B 35, 90 C 27 application of this principle to a submodular system associated with the column set of an LM-matrix is found in Murota [14], which extends the SP-decomposition of bipartite graphs due to McCormick [12]. It has been shown the principal structure is the coarsest decomposition that is finer than any decompositions induced by the CCF of the submatrix consisting of a base of the row set.

The concept of principal structure has been extended by Tomizawa–Fujishige [18] to that of a submodular function on a general (not necessarily distributive) lattice. This extension has been used in investigating the combinatorial aspects of design-variable selections in engineering [10]. It has been shown that the principal structure of a submodular function on a modular lattice associated with the row side of an LM-matrix gives the coarsest decomposition that is finer than any decompositions induced by the CCF of the submatrix consisting of a base of the column set. This characterizes the upper bound on the extent to which the discrete physical/engineering system described by an LM-matrix can be decomposed by a suitable choice of design variables.

These two results on the LM-matrices have been unified in the recent work [11], where we have treated a bipartite graph with a pair of matroids on its vertex sets, that defines "independent matchings." It has been shown that the principal structure gives the best possible upper bound on the decompositions induced by the principal partitions.

The main purpose of this paper is to understand these results in a sufficiently general framework and reveals the practical significance of the principal structure. We first deal with a submodular system and establish a first theorem characterizing its principal structure as the refinement of the principal partitions with respect to the minimal nonnegative superbases. This characterization, however, does not explain the previously known results as straightforward consequences. Then we consider a bipartite graph with a pair of polymatroid and nonnegative supermodular system with its vertex sets, that defines "Hitchcock-type independent flows," and establishes a second theorem which contains the previously known results as special cases.

The outline of this paper is as follows. Section 2 provides preliminaries on submodular systems. In Section 3 we establish the first theorem which characterizes the principal structure of a submodular system in general in terms of the principal partitions. Section 4 is devoted to the second theorem on the principal partition for "Hitchcock-type independent flows."

2. Preliminaries on Submodular Systems

This section provides preliminaries on submodular systems. See also Fujishige [6, 7] for more detail.

2.1. Polyhedra and Fundamental Operations

Let E be a finite set and $\mathcal{D} \subseteq 2^E$ be a (distributive) lattice with $\emptyset$, $E \in \mathcal{D}$. A function $f : \mathcal{D} \to \mathbb{R}$ is said to be *submodular* if it satisfies

$$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y), \quad X, Y \in \mathcal{D}.$$

For a submodular function f on $\mathcal{D}$ with $f(\emptyset) = 0$, the pair $(\mathcal{D}, f)$ is called a *submodular system* on E and f is called the *rank function* of $(\mathcal{D}, f)$.

With a submodular system $(\mathcal{D}, f)$ is associated a *submodular polyhedron* $\mathrm{P}(f)$ defined by

$$\mathrm{P}(f) = \{x \mid x \in \mathbb{R}^E,\ \forall X \in \mathcal{D} : x(X) \leq f(X)\}.$$

A vector in $\mathrm{P}(f)$ is called a *subbase* of $(\mathcal{D}, f)$. We also define

$$\mathrm{B}(f) = \{x \mid x \in \mathrm{P}(f),\ x(E) = f(E)\},$$

which is called the *base polyhedron*. A vector in $\mathrm{B}(f)$ is a maximal subbase and is called a *base* of $(\mathcal{D}, f)$.

Let $(\mathcal{D}, f)$ be a submodular system on E. A *reduction* of $(\mathcal{D}, f)$ by a vector $x \in \mathbb{R}^E$ is defined to be a submodular system $(2^E, f^x)$ whose rank function is given by

$$f^x(X) = \min\{f(Z) + x(X - Z) \mid Z \subseteq X,\ Z \in \mathcal{D}\} \quad (X \subseteq E).$$

The submodular polyhedron of the reduction is expressed as

$$\mathrm{P}(f^x) = \{y \mid y \in \mathrm{P}(f),\ y \leq x\}.$$

A *contraction* of $(\mathcal{D}, f)$ by a subbase $x \in \mathrm{P}(f)$ is a submodular system $(2^E, f_x)$ whose rank function is given by

$$f_x(X) = \min\{f(Z) - x(Z - X) \mid Z \supseteq X,\ Z \in \mathcal{D}\} \quad (X \subseteq E).$$

The base polyhedron of the contraction is expressed as

$$\mathrm{B}(f_x) = \{y \mid y \in \mathrm{B}(f),\ y \geq x\}.$$

Note that $f_x(\emptyset) = 0$ is due to the fact that $x \in \mathrm{P}(f)$.

Let F be a subset of E. Consider a submodular system $(\mathcal{D}^F, f^F)$ defined by

$$\mathcal{D}^F = \{X \mid X \in \mathcal{D},\ X \subseteq F\},$$

$$f^F(X) = f(X) \quad (X \in \mathcal{D}^F).$$

This is called the *restriction* of $(\mathcal{D}, f)$ to F, or the *reduction* by F, and is denoted by $(\mathcal{D}, f) \cdot F$. A submodular system $(\mathcal{D}_F, f_F)$ is said to be the *contraction* of $(\mathcal{D}, f)$ by F and denoted by $(\mathcal{D}, f)/F$, where

$$\mathcal{D}_F = \{X - F \mid X \in \mathcal{D},\ X \supseteq F\},$$

$$f_F(X) = f(X \cup F) - f(F) \quad (X \in \mathcal{D}_F).$$

A submodular system obtained by repeated reductions and/or contractions of $(\mathcal{D}, f)$ by subsets is called a *minor* of $(\mathcal{D}, f)$. The following well-known lemma (cf. Lemma 3.1 in [7]), which relies on the concepts of reduction and contraction by sets, will be used later.

Lemma 2.1. *Let* $(2^E, f)$ *be a submodular system. For any* $F \subseteq E$, *there exists an extreme point* x *of the base polyhedron* $\mathrm{B}(f)$ *such that* $x(F) = f(F)$.

2.2. Duality and Supermodular Systems

A function $g : \mathcal{D} \to \mathbb{R}$ is said to be supermodular if its negative is a submodular function, i.e.,

$$g(X) + g(Y) \leq g(X \cup Y) + g(X \cap Y), \quad X, Y \in \mathcal{D}.$$

The pair $(\mathcal{D}, g)$ is called a *supermodular system* if g is a supermodular function on $\mathcal{D}$ with $g(\emptyset) = 0$. Polyhedra $\mathrm{P}(g)$ and $\mathrm{B}(g)$ defined by

$$\mathrm{P}(g) = \{x \mid x \in \mathbb{R}^E,\ \forall X \in \mathcal{D} : x(X) \geq g(X)\},$$
$$\mathrm{B}(g) = \{x \mid x \in \mathrm{P}(g),\ x(E) = g(E)\}$$

are called the *supermodular polyhedron* and the *base polyhedron*, respectively. A vector in $\mathrm{P}(g)$ is called a *superbase* and a vector in $\mathrm{B}(g)$ a *base* of the supermodular system $(\mathcal{D}, g)$.

For a submodular system $(\mathcal{D}, f)$ on E, we define

$$\overline{\mathcal{D}} = \{E - X \mid X \in \mathcal{D}\},$$
$$f^\sharp(E - X) = f(E) - f(X) \quad (X \in \mathcal{D}).$$

Then the function $f^\sharp : \overline{\mathcal{D}} \to \mathbb{R}$ is supermodular, and the pair $(\overline{\mathcal{D}}, f^\sharp)$ is called the *dual supermodular system* of $(\mathcal{D}, f)$. We also call $x \in \mathrm{P}(f^\sharp)$ a *superbase* of $(\mathcal{D}, f)$. The *dual submodular system* $(\overline{\mathcal{D}}, g^\sharp)$ of a supermodular system $(\mathcal{D}, g)$ is defined in a similar way. It can be easily checked that $\mathrm{B}(f) = \mathrm{B}(f^\sharp)$ and $(f^\sharp)^\sharp = f$.

The fundamental operations on a supermodular system are defined similarly. Let $(\mathcal{D}, g)$ be a supermodular system on E. A reduction of $(\mathcal{D}, g)$ by a vector $x \in \mathbb{R}^E$ is a supermodular system $(2^E, g_x)$ whose rank function is given by

$$g_x(Y) = \max\{g(Z) + x(Y - Z) \mid Z \subseteq Y,\ Z \in \mathcal{D}\} \qquad (Y \subseteq E).$$

The supermodular polyhedron of the reduction is expressed as

$$\mathrm{P}(g_x) = \{y \mid y \in \mathrm{P}(g),\ y \geq x\}.$$

A contraction of $(\mathcal{D}, f)$ by a superbase $x \in \mathrm{P}(g)$ is a supermodular system $(2^E, g^x)$ whose rank function is given by

$$g^x(Y) = \max\{g(Z) - x(Z - Y) \mid Z \supseteq Y,\ Z \in \mathcal{D}\} \qquad (Y \subseteq E).$$

The base polyhedron of the contraction is expressed as

$$\mathrm{B}(g^x) = \{y \mid y \in \mathrm{B}(g),\ y \leq x\}.$$

The reduction and contraction by a subset are also defined similarly to those of submodular systems.

2.3. Principal Partition with Respect to a Superbase

Given a superbase $x \in \mathrm{P}(f^\sharp)$ of a submodular system $(\mathscr{D}, f)$, we define

$$\mathscr{D}(x) = \{X \mid X \in \mathscr{D},\ f(X) = f(E) - x(E - X)\},$$

which forms a distributive lattice with $E \in \mathscr{D}(x)$. Let

$$\mathscr{C} : X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_r = E$$

be a maximal chain in $\mathscr{D}(x)$. Then $\mathscr{C}$ determines a partition $\{E_0, E_1, \ldots, E_r\}$ of E, where $E_0 = X_0$ and $E_k = X_k - X_{k-1}$ for $k = 1, \ldots, r$. Furthermore a partial order $\preceq$ is defined by

$$E_k \preceq E_l \Leftrightarrow [\forall X \in \mathscr{D}(x) : X \supseteq E_l \Rightarrow X \supseteq E_k].$$

This partition of E with the partial order, which will be denoted by $\mathscr{P}(x) = (\{E_0, E_1, \ldots, E_r\}, \preceq)$, is uniquely determined independently of the choice of the maximal chain. In addition it is known as the Jordan-Hölder type theorem for submodular functions that the family of minors $(\mathscr{D}, f) \cdot X_k / X_{k-1}$ is uniquely determined no matter which maximal chain we use. The family of minors with the partial order is called the *principal partition* with respect to the superbase x.

2.4. Principal Structure of Submodular Systems

The principal structure of submodular systems is defined as follows [5].

Consider a submodular system $(\mathscr{D}, f)$ on E. Given an element $i \in E$, we denote by $D(f; i)$ the minimum element of the distributive lattice

$$\mathscr{D}(f; i) = \{Y \mid i \in Y \in \mathscr{D},\ f(Y) = \min\{f(Z) \mid i \in Z \in \mathscr{D}\}\}.$$

Since the relation $\sqsubseteq$ defined by

$$i \sqsubseteq j \Leftrightarrow i \in D(f; j)$$

is reflexive and transitive by virtue of the submodularity of f, E is decomposed into partially ordered blocks as follows. Consider the equivalence relation $\sim$ defined on E by

$$i \sim j \Leftrightarrow i \sqsubseteq j,\ j \sqsubseteq i,$$

and split E into the equivalence classes $\{F_1, \ldots, F_s\}$. A partial order $\sqsubseteq$ is induced among the equivalence classes in such a way that $F_k \sqsubseteq F_l$ iff $i \sqsubseteq j$ for $i \in F_k$ and $j \in F_l$. This decomposition, together with the partial order $\sqsubseteq$ among the blocks, is called the *principal structure* of the submodular system $(\mathscr{D}, f)$.

In other words, this concept is understood as follows.

Given an element $X \in \mathscr{D}$ we denote by $D(f; X)$ the minimum element of the sublattice

$$\mathscr{D}(f; X) = \{Y \mid X \subseteq Y \in \mathscr{D},\ f(Y) = \min\{f(Z) \mid X \subseteq Z \in \mathscr{D}\}\ \}.$$

A mapping $\psi : \mathscr{D} \to \mathscr{D}$ is said [1] to be a *closure function* if it satisfies the following three conditions:

(CL0) $\forall X \in \mathscr{D} : X \subseteq \psi(X)$.

(CL1) $\forall X, Y \in \mathscr{D} : X \subseteq Y \Rightarrow \psi(X) \subseteq \psi(Y)$.

(CL2) $\forall X \in \mathscr{D} : \psi(\psi(X)) = \psi(X)$.

Then the mapping $D(f; \cdot) : \mathscr{D} \to \mathscr{D}$ is a closure function on $\mathscr{D}$.

For a closure function ψ, it can be easily shown that $\psi(X \cap Y) = X \cap Y$ if $\psi(X) = X$ and $\psi(Y) = Y$. That is to say, the family $\{X \mid X \in \mathscr{D},\ \psi(X) = X\}$ of "closed sets" is a lower semilattice. Therefore the subset $\mathscr{K}(f)$ defined by

$$\mathscr{K}(f) = \{X \mid X \in \mathscr{D},\ D(f; X) = X\}$$

is a lower semilattice containing E. We call $\mathscr{K}(f)$ the *principal semilattice* of $(\mathscr{D}, f)$.

The family of ideals of the principal structure coincides with the minimum sublattice which contains the principal semilattice $\mathscr{K}(f)$.

3. Principal Structure and Principal Partitions

This section is devoted to a theorem which reveals an essential relation between the principal structure and the principal partitions.

The principal partition with respect to a superbase x coincides with the principal structure of the submodular system $(\mathscr{D}, f - x)$. That is,

$$\mathscr{D}(x) = \mathscr{K}(f - x), \quad \forall x \in \mathrm{P}(f^\sharp). \tag{1}$$

Hence we focus on the relation between the principal structures of $(\mathscr{D}, f)$ and $(\mathscr{D}, f - x)$, i.e., the relation between $D(f; X)$ and $D(f - x; X)$. Note, however, in the following lemma x is not supposed to be a superbase but a nonnegative vector.

Lemma 3.1. *For any $X \in \mathscr{D}$ and any $x \geq 0$,*

$$D(f; X) \subseteq D(f - x; X).$$

Proof. Fix $X \in \mathscr{D}$ and put $D_0 = D(f; X)$ and $D_x = D(f - x; X)$ for notational simplicity. To establish $D_0 \subseteq D_x$, it suffices to show

$$f(D_0) \geq f(D_0 \cap D_x).$$

By the submodularity of f and the nonnegativity of x

$$f(D_0) - f(D_0 \cap D_x) \geq f(D_0 \cup D_x) - x(D_0 \cup D_x) - f(D_x) + x(D_x),$$

where the right-hand side must be nonnegative since $X \subseteq D_0 \cup D_x$. ∎

Let $\widehat{\mathscr{B}}$ be the set of minimal nonnegative superbases of $(\mathscr{D}, f)$. Note that $\widehat{\mathscr{B}}$ coincides with the base polyhedron of a submodular system $(2^E, \widehat{f})$ whose rank function is given by

$$\widehat{f}(X) = \min\{f(Z) \mid Z \supseteq X,\ Z \in \mathscr{D}\} - \alpha \quad (X \subseteq E), \tag{2}$$

where α is the minimum value of f, i.e.,

$$\alpha = \min\{f(Y) \mid Y \in \mathscr{D}\}.$$

Then we have the following lemma.

Lemma 3.2. *For any $X \in \mathscr{D}$, there exists a minimal nonnegative superbase $x \in \widehat{\mathscr{B}}$ of $(\mathscr{D}, f)$ such that*

$$D(f; X) = D(f - x; X).$$

Proof. Fix $X \in \mathscr{D}$ and put $D_0 = D(f; X)$ for notational simplicity. By the definitions of $\widehat{f}$ and $D(f; X)$, it is clear that $\widehat{f}(D_0) = f(D_0) - \alpha$. It follows from Lemma 2.1 that there exists $x \in \widehat{\mathscr{B}}$ such that $x(D_0) = f(D_0) - \alpha$. Note that $x(Z) \leq \widehat{f}(Z) \leq f(Z) - \alpha$ for any $Z \in \mathscr{D}$. Hence we have

$$f(D_0) - x(D_0) = \min\{f(Z) - x(Z) \mid X \subseteq Z \in \mathscr{D}\},$$

which implies $D_0 \supseteq D(f - x; X)$. Since $D_0 \subseteq D(f - x; X)$ is proven in Lemma 3.1, we may conclude that $D_0 = D(f - x; X)$. ∎

We are now ready to establish a relation between the principal structure and principal partitions. It follows from Lemma 3.1 that $D(f - x; X) = X$ implies $D(f; X) = X$. Hence,

$$\mathscr{K}(f - x) \subseteq \mathscr{K}(f)$$

holds for any nonnegative vector x. On the other hand, from Lemma 3.2, $X = D(f; X)$ implies the existence of a nonnegative base x such that $X = D(f - x; X)$. That is to say,

$$\mathscr{K}(f) \subseteq \bigcup_{x \in \widehat{\mathscr{B}}} \mathscr{K}(f - x).$$

Thus we have

$$\mathscr{K}(f) = \bigcup_{x \in \widehat{\mathscr{B}}} \mathscr{K}(f - x),$$

which, together with (1), implies the following theorem. See § 2.3 for the definition of $\mathscr{D}(x)$.

Theorem 3.3. *For a submodular system $(\mathscr{D}, f)$,*

$$\mathscr{K}(f) = \bigcup_{x \in \widehat{\mathscr{B}}} \mathscr{D}(x),$$

where $\widehat{\mathscr{B}}$ is the set of minimal nonnegative superbases of $(\mathscr{D}, f)$. ∎

As is evident from the proof, we may restrict the nonnegative bases x in Lemma 3.2 to the extreme points of $\widehat{\mathscr{B}}$. Therefore we have the following corollary.

Corollary 3.4. *For a submodular system $(\mathscr{D}, f)$,*

$$\mathscr{K}(f) = \bigcup_{x \in \dot{\mathscr{B}}} \mathscr{D}(x),$$

where $\dot{\mathscr{B}}$ designates the set of extreme points of the polyhedron which consists of the minimal nonnegative superbases of $(\mathscr{D}, f)$. ∎

With the aid of Birkhoff's representation theorem [1], Corollary 3.4 is translated to Corollary 3.5 in terms of partitions and partial order. Let Π be the collection of the pairs $\{\pi, \preceq\}$ of a partition π of E and a partial order $\preceq$ among the blocks of π. A partial order, denoted by $\preccurlyeq$, is introduced on Π in such a way that $\mathscr{P}' \preccurlyeq \mathscr{P}''$ iff $\mathscr{P}' = (\{F'_k\}, \preceq')$ is a refinement of $\mathscr{P}'' = (\{F''_l\}, \preceq'')$, i.e., (i) $\{F'_k\}$ is a refinement of $\{F''_l\}$ as a partition and (ii) $F'_{k_h} \subseteq F''_{l_h}$ $(h=1,2)$ and $F'_{k_1} \preceq' F'_{k_2}$ implies $F''_{l_1} \preceq'' F''_{l_2}$. It is easy to see that the partially ordered set $(\Pi, \preccurlyeq)$ forms a lattice. Then we have the following corollary.

Corollary 3.5. *Let $\mathscr{P}_{\mathrm{PS}}$ be the principal structure of a submodular system $(\mathscr{D}, f)$. Then*

$$\mathscr{P}_{\mathrm{PS}} = \bigwedge_{x \in \dot{\mathscr{B}}} \mathscr{P}(x),$$

where $\bigwedge$ designates the meet operation in the lattice Π. ∎

Example 3.6. Consider a function f on $E = \{e_1, e_2, e_3\}$ defined by

$$f(X) = \begin{cases} 0 & (X = \emptyset), \\ 1 & (|X| = 1 \text{ or } X = E), \\ 2 & (|X| = 2). \end{cases}$$

Then $(2^E, f)$ is a submodular system. The base polyhedron is an equilateral triangle with vertices $(1,1,-1)$, $(1,-1,1)$ and $(-1,1,1)$. The function $\widehat{f}$ is given by

$$\widehat{f}(X) = \begin{cases} 0 & (X = \emptyset), \\ 1 & (X \neq \emptyset). \end{cases}$$

Therefore $\dot{\mathscr{B}} = \{x_1, x_2, x_3\}$ where $x_1 = (1,0,0)$, $x_2 = (0,1,0)$ and $x_3 = (0,0,1)$. The principal partitions with respect to theses bases are as follows: $\mathscr{D}(x_1) = \{\emptyset, \{e_1\}, E\}$, $\mathscr{D}(x_2) = \{\emptyset, \{e_2\}, E\}$ and $\mathscr{D}(x_3) = \{\emptyset, \{e_3\}, E\}$. On the other hand, the principal semilattice is $\mathscr{K}(f) = \{\emptyset, \{e_1\}, \{e_2\}, \{e_3\}, E\}$. We can see that the principal semilattice is the union of the principal partitions with respect to the bases in $\dot{\mathscr{B}}$. ∎

Example 3.7. Let f be a function on $E=\{e_1, e_2\}$ defined by

$$f(X) = \begin{cases} -1 & (X = \{e_1\}), \\ 0 & (X = \emptyset \text{ or } X = E), \\ 2 & (X = \{e_2\}). \end{cases}$$

The base polyhedron $\mathrm{B}(f)$ is a closed line segment connecting $(-1,1)$ to $(-2,2)$. This submodular system $(2^E, f)$ has the only minimal nonnegative superbase $x = (0,1)$. The function $\widehat{f}$ is obtained as

$$\widehat{f}(X) = \begin{cases} 0 & (X = \emptyset \text{ or } X = \{e_1\}), \\ 1 & (X = \{e_2\} \text{ or } X = E). \end{cases}$$

Note that x is not a base of $(2^E, f)$ but of $(2^E, \widehat{f})$. The principal partition of $(2^E, f)$ with respect to the superbase x is given by $\mathscr{D}(x) = \{\{e_1\}, E\}$, which coincides with the principal semilattice $\mathscr{K}(f) = \{\{e_1\}, E\}$. ∎

4. Hitchcock-type Independent Flows

In this section, we introduce the concept of a *Hitchcock-type independent flow*, which is a polymatroidal extension of the *Hitchcock-type transportation*, and apply Theorem 3.3 to its analysis.

4.1. Definitions

A *polymatroid* $\mathbf{P} = (S, \rho)$ is a submodular system $(2^S, \rho)$ whose rank function ρ is monotone nondecreasing, i.e.,

$$\forall X, Y \subseteq S : X \subseteq Y \Rightarrow \rho(X) \leq \rho(Y).$$

It can be easily checked that a submodular system $(2^S, \rho)$ is a polymatroid iff its dual supermodular system has a nonnegative rank function. Note that a nonnegative supermodular function is necessarily monotone nondecreasing.

Consider a bipartite graph $G = (S, T; A)$ with a polymatroid $\mathbf{P} = (S, \rho)$ on the vertex set S and a supermodular system $(2^T, \sigma)$ with σ nonnegative on the other vertex set T. This network $\mathscr{H}=(G,\mathbf{P},\mathbf{Q})$ defines Hitchcock-type independent flows, where $\mathbf{Q}=(T,\sigma)$ denotes the nonnegative supermodular system $(2^T,\sigma)$.

For a *flow* $\varphi: A \to \mathbb{R}$, which is a nonnegative function on the arc set A, we define $\partial^+\varphi \in \mathbb{R}^S$ and $\partial^-\varphi \in \mathbb{R}^T$ by

$$\partial^+\varphi(v) \equiv \sum_{a\in\delta v} \varphi(a) \qquad (v \in S),$$
$$\partial^-\varphi(v) \equiv \sum_{a\in\delta v} \varphi(a) \qquad (v \in T),$$

where δv denotes the set of the arcs incident to v. A flow φ is said to be a *feasible flow* or a Hitchcock-type independent flow in $\mathscr{H}$ if it satisfies $\partial^+\varphi \in \mathrm{P}(\rho)$ and $\partial^-\varphi \in \mathrm{P}(\sigma)$.

Since σ is nonnegative, the dual submodular system $(2^T,\sigma^\sharp)$ is a polymatroid, which we denote by $\mathbf{Q}^\sharp = (T,\sigma^\sharp)$. Then the existence of a feasible flow in $\mathscr{H} = (G,\mathbf{P},\mathbf{Q})$ is equivalent to that of an independent flow of value $\sigma(T)$ in the network $\mathscr{N} = (G,\mathbf{P},\mathbf{Q}^\sharp)$. Applying the maximum-independent-flow minimum-cut theorem [4] to the network $\mathscr{N}$, we obtain the following lemma, which can be derived also from the discrete separation theorem [3]. Recall that a *cover* of G is a pair (U,W) of $U\subseteq S$ and $W\subseteq T$ such that no edges exist between $S-U$ and $T-W$.

Lemma 4.1. *The network $\mathscr{H}=(G,\mathbf{P},\mathbf{Q})$ has a feasible flow iff $\rho(U)\geq\sigma(T-W)$ for all covers (U,W) of G.* ∎

According to the general principle expounded in Iri [9], the *principal partition* of $\mathscr{H}$ can be introduced as follows. Let $\mathcal{U}$ be the distributive lattice which consists of all the covers of G. The operations in $\mathcal{U}$ are defined by

$$(U_1,W_1)\vee(U_2,W_2) = (U_1\cap U_2, W_1\cup W_2),$$
$$(U_1,W_1)\wedge(U_2,W_2) = (U_1\cup U_2, W_1\cap W_2).$$

The rank of a cover (U,W) is $\rho(U)-\sigma(T-W)$, and the set of the minimum-rank covers forms a distributive lattice $\mathcal{U}^*$, which yields a direct sum decomposition of the network $\mathscr{H}$. This decomposition will be called the *principal partition* of $\mathscr{H}$, and gives useful informations on the structure of Hitchcock-type independent flows. See Appendix for the detail.

We now introduce a submodular function $f: 2^S \to \mathbb{R}$ defined by

$$f(X) = \sigma^\sharp(\Gamma_T(X)) - \rho^\sharp(X) \quad (X \subseteq S), \tag{3}$$

where $\Gamma_T(X)$ denotes the set of the vertices adjacent to a vertex in X, i.e.,

$$\Gamma_T(X) = \{j \mid j\in T,\ \exists i \in X : (i,j)\in A\}.$$

Note that $f(\emptyset)=0$.

For the sake of simplicity we assume that the bipartite graph G has no isolated vertex. Hence $\Gamma_T(S)=T$ and $f(S)=\sigma(T)-\rho(S)$. It can be easily shown that

$$\min\{\rho(U)-\sigma(T-W) \mid (U,W)\in\mathcal{U}\} = \min\{f(X) \mid X\subseteq S\} - f(S). \tag{4}$$

Thus Lemma 4.1 is rephrased as follows.

Lemma 4.2. *The network $\mathcal{H}=(G,\mathbf{P},\mathbf{Q})$ has a feasible flow iff*

$$\min\{f(X) \mid X\subseteq S\} = f(S).$$ ∎

Let $\mathcal{L}$ be the family of the minimizers of f. Then $\mathcal{L}$ forms a sublattice of 2^S. Note that $\mathcal{L}$ coincides with the distributive lattice induced from the principal partition, i.e.,

$$\mathcal{L}=\{S-U \mid \exists W\subseteq T:(U,W)\in\mathcal{U}^*\}.$$

We also define a supermodular function $g:2^T\to\mathbb{R}$ by

$$g(Y)=\sigma(Y)-\rho(\Gamma_S(Y)) \quad (Y\subseteq T),$$

where

$$\Gamma_S(Y)=\{i \mid i\in S,\ \exists j\in Y:(i,j)\in A\}.$$

Note that $g(\emptyset)=0$. It can be easily shown that

$$\min\{\rho(U)-\sigma(T-W) \mid (U,W)\in\mathcal{U}\} = -\max\{g(Y) \mid Y\subseteq T\}. \tag{5}$$

Hence Lemma 4.1 is also rephrased in terms of the supermodular function g.

Lemma 4.3. *The network $\mathcal{H}=(G,\mathbf{P},\mathbf{Q})$ has a feasible flow iff*

$$\max\{g(Y) \mid Y\subseteq T\}=0.$$ ∎

4.2. Result

This section discusses the principal structure of the submodular system $(2^S,f)$ defined by (3). Our main result (Theorem 4.6) shows that the principal structure gives the best-possible upper bound on the extent to which S is decomposed by the principal partition of a feasible network obtained by an augmentation of the supply or a reduction of the demand.

When the network is infeasible, we have to augment the supply so as to meet the demand. As an augmentation of the supply it seems natural to consider a translation of the polymatroid $\mathbf{P}$ by a nonnegative vector. The following lemma shows how we should augment the supply to get a feasible network.

Lemma 4.4. *For a nonnegative vector* $x \in \mathbb{R}^S$, *let* $\hat{\mathbf{P}}$ *be the polymatroid* $(S, \rho + x)$. *Then the network* $\hat{\mathcal{H}} = (G, \hat{\mathbf{P}}, \mathbf{Q})$ *has a feasible flow iff* $x \in \mathrm{P}(f^\sharp)$.

Proof. It follows from Lemma 4.2 that $\hat{\mathcal{H}} = (G, \hat{\mathbf{P}}, \mathbf{Q})$ is feasible iff

$$f(X) - x(X) \geq f(S) - x(S), \quad \forall X \subseteq S,$$

which is equivalent to $x \in \mathrm{P}(f^\sharp)$. ∎

Hence Theorem 3.3 means that the principal structure of the submodular function f gives the coarsest decomposition of S that is finer than the principal partition of any feasible network obtained by a minimal augmentation of the supply.

Another way to make the network feasible is to reduce the demand. This is represented by a reduction of the nonnegative supermodular system $\mathbf{Q}$ by a nonnegative vector y and a translation by $-y$. We denote by $\sigma_{\langle y \rangle}$ the rank function $\sigma_y - y$ of the supermodular system obtained by this operation (see § 2.1 for the notation of σ_y). That is,

$$\sigma_{\langle y \rangle}(Y) = \max\{\sigma(Z) - y(Z) \mid Z \subseteq Y\} \quad (Y \subseteq T). \tag{6}$$

Note that $\sigma_{\langle y \rangle}$ is still a nonnegative supermodular function. The following lemma tells that the supermodular system $(2^T, g)$ shows how we should reduce the demand to get a feasible network.

Lemma 4.5. *Denote* $(T, \sigma_{\langle y \rangle})$ *by* $\mathbf{Q}'$. *Then the network* $\mathcal{H}' = (G, \mathbf{P}, \mathbf{Q}')$ *has a feasible flow iff* $y \in \mathrm{P}(g)$.

Proof. It follows from Lemma 4.3 that $\mathcal{H}' = (G, \mathbf{P}, \mathbf{Q}')$ is feasible iff

$$\sigma_{\langle y \rangle}(Y) - \rho(\Gamma_S(Y)) \leq 0, \quad \forall Y \subseteq T.$$

This is equivalent to

$$\sigma(Z) - \min\{\rho(\Gamma_S(Y)) \mid Y \supseteq Z\} \leq y(Z), \quad \forall Z \subseteq T.$$

From the monotonicity of ρ this is further equivalent to

$$\sigma(Z) - \rho(\Gamma_S(Z)) \leq y(Z), \quad \forall Z \subseteq T,$$

which holds iff $y \in \mathrm{P}(g)$. ∎

Let $\mathcal{B}'$ be the set of minimal nonnegative superbases of $(2^T, g)$. This is the set of minimal nonnegative vectors y by which reductions of the demand make the network feasible.

Let us consider the principal partition of the network $\mathcal{H}' = (G, \mathbf{P}, \mathbf{Q}')$. We define $\mathcal{L}(y)$ to be the sublattice of 2^S induced from the principal partition of $\mathcal{H}' = (G, \mathbf{P}, \mathbf{Q}')$. That is,

$$\mathcal{L}(y) = \{S - U \mid \exists W \subseteq T : (U, W) \in \mathcal{U}^*(y)\},$$

where $\mathcal{U}^*(y)$ denotes the family of minimum-rank covers of $\mathcal{H}' = (G, \mathbf{P}, \mathbf{Q}')$. Note that $\mathcal{L}(y)$ is the set of minimizers of the submodular function $f_{\langle y\rangle}$ given by

$$(7) \qquad f_{\langle y\rangle}(X) = \sigma_{\langle y\rangle}{}^{\sharp}(\Gamma_T(X)) - \rho^{\sharp}(X) \quad (X \subseteq S).$$

As to the principal structure of the submodular system $(2^S, f)$, we have the following theorem whose proof is postponed to Section 4.3.

Theorem 4.6. *Let $\mathcal{K}(f)$ be the principal semilattice of f. Then we have*

$$\mathcal{K}(f) = \bigcup_{x \in \widehat{\mathcal{B}}} \mathcal{D}(x) = \bigcup_{y \in \mathcal{B}'} \mathcal{L}(y),$$

where $\widehat{\mathcal{B}}$ and $\mathcal{B}'$ denote the set of minimal nonnegative superbases of $(2^S, f)$ and $(2^T, g)$, respectively.

Note that the first equality is shown in Theorem 3.3. It will become obvious from the proof that we may restrict $y \in \mathcal{B}'$ to be integral if the rank functions ρ and σ are integer-valued.

4.3. Proof

We begin with a fundamental property of the parametrized family of supermodular functions $\sigma_{\langle\cdot\rangle}(\cdot)$ defined by (6).

Lemma 4.7. *For any $Y_1, Y_2 \subseteq T$ and $y_1, y_2 \in \mathbb{R}^T$ it holds that*

$$\sigma_{\langle y_1\rangle}(Y_1) + \sigma_{\langle y_2\rangle}(Y_2) \le \sigma_{\langle y_1 \vee y_2\rangle}(Y_1 \cap Y_2) + \sigma_{\langle y_1 \wedge y_2\rangle}(Y_1 \cup Y_2),$$

where $y_1 \vee y_2 \in \mathbb{R}^T$, $y_1 \wedge y_2 \in \mathbb{R}^T$ such that $(y_1 \vee y_2)(j) = \max\{y_1(j), y_2(j)\}$ and $(y_1 \wedge y_2)(j) = \min\{y_1(j), y_2(j)\}$ for $j \in T$.

Proof. Suppose $Z_1 \subseteq Y_1$ and $Z_2 \subseteq Y_2$ attain $\sigma_{\langle y_1\rangle}(Y_1) = \sigma(Z_1) - y_1(Z_1)$ and $\sigma_{\langle y_2\rangle}(Y_2) = \sigma(Z_2) - y_2(Z_2)$, respectively. Using the fact that $Z_1 \cap Z_2 \subseteq Y_1 \cap Y_2$ and $Z_1 \cup Z_2 \subseteq Y_1 \cup Y_2$, we have

$$\begin{aligned}
\sigma_{\langle y_1\rangle}(Y_1) + \sigma_{\langle y_2\rangle}(Y_2) &= \sigma(Z_1) + \sigma(Z_2) - y_1(Z_1) - y_2(Z_2) \\
&\le \sigma(Z_1 \cap Z_2) - (y_1 \vee y_2)(Z_1 \cap Z_2) + \sigma(Z_1 \cup Z_2) - (y_1 \wedge y_2)(Z_1 \cup Z_2) \\
&\le \max\{\sigma(Z) - (y_1 \vee y_2)(Z) \mid Z \subseteq Y_1 \cap Y_2\} \\
&\quad + \max\{\sigma(Z) - (y_1 \wedge y_2)(Z) \mid Z \subseteq Y_1 \cup Y_2\}.
\end{aligned}$$ ∎

As a consequence of this lemma, we have the following corollary on $f_{\langle\cdot\rangle}(\cdot)$ defined by (7).

Corollary 4.8. *For any $X_1, X_2 \subseteq S$ and $y_1, y_2 \in \mathbb{R}^T$ it holds that*

$$f_{\langle y_1 \rangle}(X_1) + f_{\langle y_2 \rangle}(X_2) \geq f_{\langle y_1 \wedge y_2 \rangle}(X_1 \cap X_2) + f_{\langle y_1 \vee y_2 \rangle}(X_1 \cup X_2).$$

Proof. It follows from Lemma 4.7 that

$$\begin{aligned} &\sigma_{\langle y_1 \rangle}{}^{\sharp}(\Gamma_T(X_1)) + \sigma_{\langle y_2 \rangle}{}^{\sharp}(\Gamma_T(X_2)) \\ &\geq \sigma_{\langle y_1 \wedge y_2 \rangle}{}^{\sharp}(\Gamma_T(X_1 \cap X_2)) + \sigma_{\langle y_1 \vee y_2 \rangle}{}^{\sharp}(\Gamma_T(X_1 \cup X_2)), \end{aligned}$$

which, together with the supermodularity of $\rho^{\sharp}$, completes the proof. ∎

This corollary plays an essential role in proving the following lemma similar to Lemma 3.1.

Lemma 4.9. *For any $X \subseteq S$ and any $y \geq 0$ we have*

$$D(f; X) \subseteq D(f_{\langle y \rangle}; X).$$

Proof. Fix $X \in \mathscr{D}$ and put $D_0 = D(f; X)$ and $D_y = D(f_{\langle y \rangle}; X)$ for notational simplicity. To establish $D_0 \subseteq D_y$, it suffices to show

$$f(D_0) \geq f(D_0 \cap D_y).$$

Since y is nonnegative, we have $y \vee 0 = y$ and $y \wedge 0 = 0$. Then it follows from Corollary 4.8 that

$$f(D_0) - f(D_0 \cap D_y) \geq f_{\langle y \rangle}(D_0 \cup D_y) - f_{\langle y \rangle}(D_y),$$

where the right-hand side must be nonnegative since $X \subseteq D_0 \cup D_y$. ∎

Consider the reduction of $(2^T, g)$ by the zero vector. The rank function is given by

$$g_0(Y) = \max\{g(Z) \mid Z \subseteq Y\} \quad (Y \subseteq T).$$

The base polyhedron of $(2^T, g)$ coincides with the set $\mathscr{B}'$ of minimal nonnegative superbases of $(2^T, g)$. Then we have the following lemma.

Lemma 4.10. *For any $x \in \widehat{\mathscr{B}}$ there exists $y \in \mathscr{B}'$ such that $\mathscr{D}(x) \subseteq \mathscr{L}(y)$.*

Proof. Fix $x \in \widehat{\mathscr{B}}$ and denote the polymatroid $(S, \rho + x)$ by $\hat{\mathbf{P}}$. Recall that $\widehat{\mathscr{B}}$ coincides with the base polyhedron of $(2^S, \widehat{f})$ where $\widehat{f}$ is defined by (2).

As is shown in Lemma 4.4, the network $\hat{\mathscr{H}} = (G, \hat{\mathbf{P}}, \mathbf{Q})$ has a feasible flow $\hat{\varphi}$. Decrease the flow monotonously to get a flow φ such that $\partial^+ \varphi = \partial^+ \hat{\varphi} - x$ and $\varphi \leq \hat{\varphi}$. Put $y = \partial^- \hat{\varphi} - \partial^- \varphi$, and then φ is a feasible flow in the network $\mathscr{H}' = (G, \mathbf{P}, \mathbf{Q}')$.

Since $y(T)=x(S)=\widehat{f}(S)$, it follows from (4) and (5) that $y(T)=g_0(T)$, and thus $y\in\mathscr{B}'$.

We now claim that $X\in\mathscr{L}(y)$ if $X\in\mathscr{D}(x)$. Since the network $\mathscr{H}'=(G,\mathbf{P},\mathbf{Q}')$ is feasible, it suffices to show that $f_{\langle y\rangle}(X)\leq f_{\langle y\rangle}(S)$. A direct calculation using (6) and (7) shows that

$$f_{\langle y\rangle}(X)=\sigma_{\langle y\rangle}(T)-\sigma(T)+y(T)-\rho^\sharp(X)+\min\{\sigma^\sharp(Y)-y(Y)\mid Y\supseteq\Gamma_T(X)\}.$$

Hence, noting $f(S)=\widehat{f}(S)+\min\{f(Z)\mid Z\subseteq S\}$, we have

$$\begin{aligned}f_{\langle y\rangle}(X)-f_{\langle y\rangle}(S)&=\min\{\sigma^\sharp(Y)-y(Y)\mid Y\supseteq\Gamma_T(X)\}-\rho^\sharp(X)-\min\{f(Z)\mid Z\subseteq S\}\\&\leq f(X)-y(\Gamma_T(X))-\min\{f(Z)\mid Z\subseteq S\}\leq f(X)-x(X)-\min\{f(Z)\mid Z\subseteq S\}=0,\end{aligned}$$

where the last equality follows from $x\in\widehat{\mathscr{B}}$ and $X\in\mathscr{D}(x)$. ∎

It follows from Lemma 4.9 that $\mathscr{L}(y)\subseteq\mathscr{K}(f)$ for any nonnegative vector y. On the other hand, Theorem 3.3 and Lemma 4.10 show that

$$\mathscr{K}(f)=\bigcup_{x\in\widehat{\mathscr{B}}}\mathscr{D}(x)\subseteq\bigcup_{y\in\mathscr{B}'}\mathscr{L}(y).$$

Thus we have proved Theorem 4.6.

4.4. Discussion

The previously known result [11] on the principal structure for independent matchings can be understood from the present result as follows. Suppose both $\mathbf{P}=(S,\rho)$ and $\mathbf{Q}^\sharp=(T,\sigma^\sharp)=(T,\varsigma)$ are matroidal polymatroids, i.e., $\rho(X)\leq|X|$ for all $X\subseteq S$ and $\varsigma(Y)\leq|Y|$ for all $Y\subseteq T$. Let $\mathbf{M}=(T,\mu)$ be the union matroid of (T,τ) and (T,ς^*), where (T,τ) is induced from $\mathbf{P}$ by G and (T,ς^*) is the dual matroid of $\mathbf{Q}^\sharp$. The rank functions are given by

$$\begin{aligned}&\mu(Y)=\min\{\tau(Z)+\varsigma^*(Z)-|Z|\mid Z\subseteq Y\}+|Y|\quad(Y\subseteq T),\\&\tau(Y)=\min\{\rho(\Gamma_S(Z))-|Z|\mid Z\subseteq Y\}+|Y|\quad(Y\subseteq T),\\&\varsigma^*(Y)=|Y|-\sigma(Y)\quad(Y\subseteq T).\end{aligned}$$

Then a simple calculation derives

$$\mu(Y)=|Y|-g_0(Y),$$

which implies that the integral bases of $(2^T,g_0)$ correspond to the characteristic vectors of the cobases of $\mathbf{M}$. Suppose $y\in\mathscr{B}'$ is the characteristic vector of the cobase K of $\mathbf{M}$. Then we have

$$\sigma_{\langle y\rangle}(Y)=\max\{\sigma(Z)-|Z\cap K|\mid Z\subseteq Y\}=\sigma(Y-K).$$

Hence we have the following corollary to Theorem 4.6.

Corollary 4.11. *Denote by $\mathscr{L}[B]$ the sublattice of 2^S induced by the principal partition of $\mathscr{H}_B = (G_B, \mathbf{P}, \mathbf{Q} \cdot B)$, where G_B is the induced subgraph of G on the vertex set $S \cup B$. Let $\mathscr{B}$ be the base family of $\mathbf{M}$ and $\mathcal{K}(f)$ be the principal semilattice of $(2^S, f)$. Then we have* $\mathcal{K}(f) = \bigcup_{B \in \mathscr{B}} \mathscr{L}[B]$. ∎

This is tantamount to the main result (Theorem 3) in [11], since

$$\sigma_{\langle y \rangle}{}^{\sharp}(J) = \sigma^{\sharp}(J \cup K) - \sigma^{\sharp}(K)$$

is the rank function of the contraction of $\mathbf{Q}^{\sharp}$ by K.

5. Conclusion

A concrete meaning of the principal structure of submodular systems is discussed. It has been shown that the principal structure is the coarsest decomposition that is finer than any decomposition induced by the principal partition with respect to a minimal nonnegative superbase. This result reflects that the principal structure is essentially concerned with such concepts as nonnegativity, whereas most materials in submodular systems theory are invariant with respect to translations.

Appendix

This appendix explains the principal partition of the network $\mathscr{H}$. See § 4.1 for the notations.

Let

$$\mathscr{C}^* : (U_0, W_0) \gneqq (U_1, W_1) \gneqq \cdots \gneqq (U_r, W_r)$$

be a maximal chain in $\mathcal{U}^*$, and put

$$\begin{aligned} S_0 &= S - U_0, & T_0 &= W_0, & \\ S_k &= U_{k-1} - U_k, & T_k &= W_k - W_{k-1} & \text{for } k = 1, \ldots, r, \\ S_\infty &= U_r, & T_\infty &= T - W_r. & \end{aligned}$$

Note that the network $\mathscr{H}$ has a feasible flow iff $S_\infty = T_\infty = \emptyset$.

Let $G_k = (S_k, T_k; A_k)$ be the induced subgraph of G for $k = 0, 1, \ldots, r, \infty$. Polymatroidal structures are also induced by

$$\begin{aligned} \mathbf{P}_0 &= \mathbf{P}/U_0, & \mathbf{Q}_0 &= \mathbf{Q} \times W_0, & \\ \mathbf{P}_k &= \mathbf{P} \cdot U_{k-1}/U_k, & \mathbf{Q}_k &= \mathbf{Q} \times W_k \backslash W_{k-1}, & \text{for } k = 1, \ldots, r, \\ \mathbf{P}_\infty &= \mathbf{P} \cdot U_r, & \mathbf{Q}_\infty &= \mathbf{Q} \backslash W_r, & \end{aligned}$$

where $\mathbf{Q} \times W$ denotes the contraction by $T - W$ and $\mathbf{Q} \backslash W$ denotes the restriction to $T - W$. (See § 2.1 for the definitions of $\mathbf{P}/U$ and $\mathbf{P} \cdot U$.)

This decomposition of the network $\mathcal{H}$ into components $\mathcal{H}_k = (G_k, \mathbf{P}_k, \mathbf{Q}_k)$ will be called the principal partition of $\mathcal{H}$, and gives useful informations on the structure of Hitchcock-type independent flows. For example, feasible flows never use the edges that connect different components. Therefore we may compute a "minimum-cost feasible flow" of a large-scale network $\mathcal{H}$ by solving the optimization problem in each component $\mathcal{H}_k$ independently.

Suppose the rank functions are given by $\rho(X) = |X|$ and $\sigma(Y) = |Y|$. In this case the principal partition of $\mathcal{H}$ is nothing but the Dulmage-Mendelsohn decomposition [2] of the bipartite graph G. Thus the principal partition is an extension of the Dulmage-Mendelsohn decomposition.

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Satoru Iwata

Research Institute
for Mathematical Sciences,
Kyoto University,
Kyoto 606, Japan
iwata@kurims.kyoto-u.ac.jp